Clones and Clonoids

Patrick Wynne

Charles University Department of Algebra

24 March 2026

Part 1: Clones

Definition of a Clone

Definition

A **clone** on a set A is a set $\mathcal{C}$ of finitary functions on A that

- is closed under composition, and
- contains all projection maps π_i^n.

$$\pi_i^n \colon A^n \to A, \quad (x_1, \ldots, x_n) \mapsto x_i.$$

Let $C^{(n)} = C \cap A^{A^n}$, the set of n-ary functions in C.
Composition: C clone, $f \in C^{(n)}, g_1, \ldots, g_n \in C^{(k)}$.
Then the composition $f \circ (g_1, \ldots, g_n) \in C^{(k)}$, where

$$f \circ (g_1, \ldots, g_n)(x_1, \ldots, x_k) = f(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k)).$$

Examples?
What are they good for?

Examples

- The clone of all functions on a set, $\mathcal{O}_A := \bigcup_{n\in\mathbb{N}} A^{A^n}$.
- $\mathcal{J}_A :=$ the clone of projections on A.
- Clones of linear functions:

$$f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n.$$

- Clones of monotone functions:

f is monotone if

$$a_1 \leq b_1, \ldots, a_n \leq b_n \implies f(a_1, \ldots, a_n) \leq f(b_1, \ldots, b_n).$$

If $f : A^n \to A$ and $g_1, \ldots, g_n \colon A^k \to A$ are monotone then

$$f(g_1(a_1, \ldots, a_k), \ldots, g_n(a_1, \ldots, a_k)) \leq f(g_1(b_1, \ldots, b_k), \ldots, g_n(b_1, \ldots, b_k))$$

whenever $a_1 \leq b_1, \ldots, a_k \leq b_k$.

Clones and Algebraic Structures

Let $\mathbf{A} = (A; f_1, f_2, \dots)$ be an algebraic structure (algebra).

$f_i \colon A^{n_i} \to A$ basic operation of $\mathbf{A}$.

A function $t \colon A^n \to A$ is a *term function* of $\mathbf{A}$
if t is obtained by composition from $f_1, f_2, \dots$ and projections.

Ex: If $\mathbf{A} = (\{0,1\}; \vee, \wedge)$ then

$$t(x,y,z) = (x \vee y) \wedge (x \vee z)$$

is a term function of $\mathbf{A}$.
$\mathsf{Clo}(\mathbf{A})$ is the term clone of $\mathbf{A}$.
It's the smallest clone on A containing $f_1, f_2, \dots$.

This is essentially the only example... Every clone on a set A is the term clone of some algebra $(A, f_1, f_2, \dots)$.

Clones of Term Functions

Examples:

$\mathbf{A} = (\mathbb{Z}; +, -, \cdot, 0, 1)$,

$$\mathsf{Clo}(\mathbf{A}) = \mathbb{Z}[x_1, x_2, \dots].$$

$\mathbf{B} = (\{0, 1\}; +, -, 0)$,

$$\mathsf{Clo}(\mathbf{B}) = \{a_1 x_1 + \cdots + a_n x_n \mid n \in \mathbb{N}, a_1, \dots, a_n \in \{0, 1\}\}.$$

$\mathbf{C} = (\{0, 1\}; \wedge, \vee, \neg)$,

$$\mathsf{Clo}(\mathbf{C}) = \mathcal{O}_{\{0,1\}}.$$

$\mathbf{D} = (\{0, 1\}; +, -, 0, \cdot, 1)$,

$$\mathsf{Clo}(\mathbf{D}) = \mathcal{O}_{\{0,1\}}$$

Term Equivalence

$\mathsf{Clo}(\mathbf{C}) = \mathsf{Clo}(\mathbf{D})$ even though $\mathbf{C} \neq \mathbf{D}$.

Another example:

$$\mathsf{Clo}(\{0,1\}; +) = \mathsf{Clo}(\{0,1\}; x + y + z, 0).$$

Algebras are "essentially the same" if they are *term equivalent*,
i.e. they have the same term clone.

How many "essentially different" algebras on a finite set?
i.e. how many algebras are there up to term equivalence?
$|A| = 1$: Just one clone: $\mathcal{J}_A = \mathcal{O}_A$.

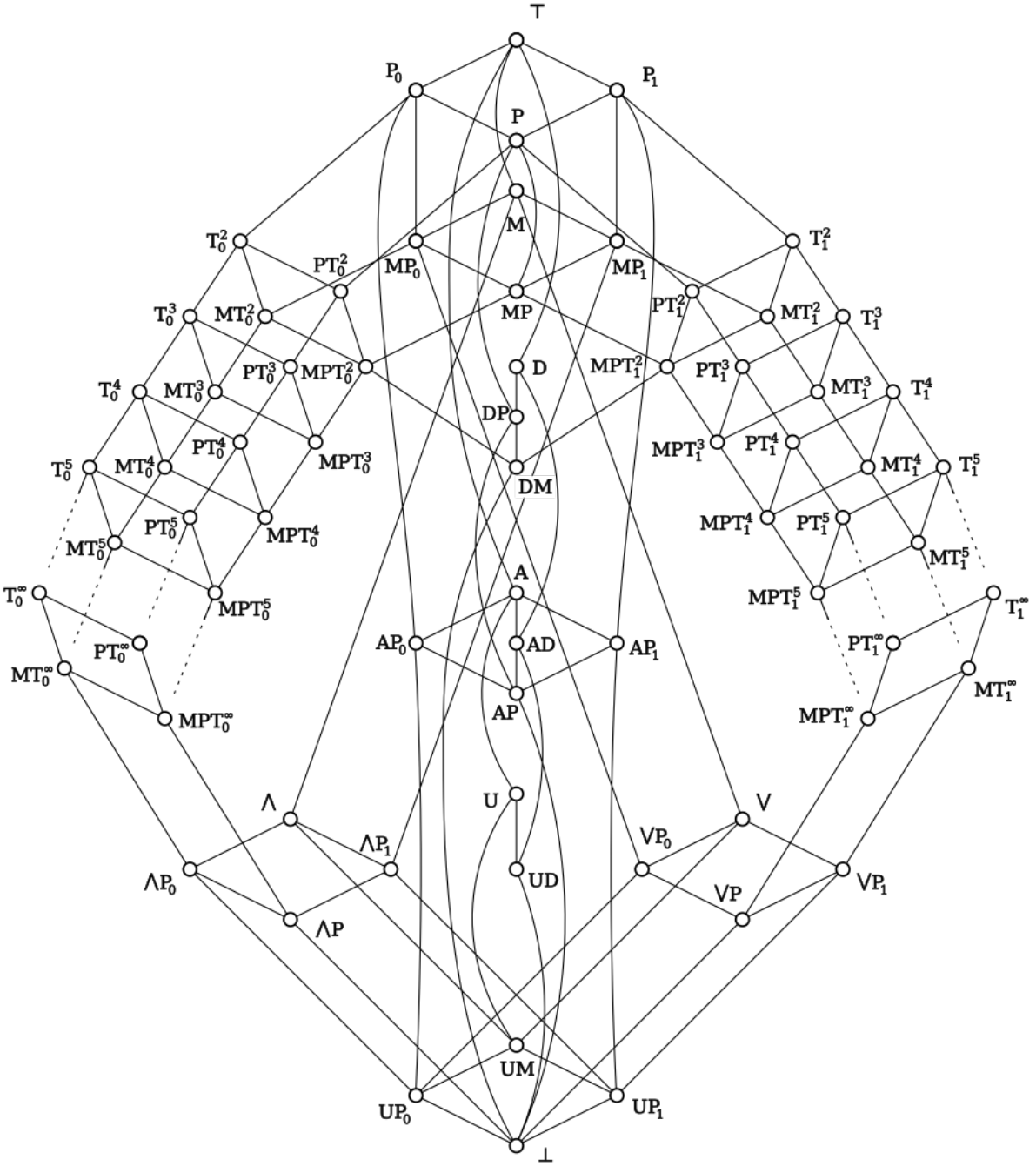
$A = \{0,1\}$:

$$C \subseteq \bigcup_{n \in \mathbb{N}} A^{A^n}$$

At most, $2^{\aleph_0}$ (continuum many) clones on $\{0,1\}$.

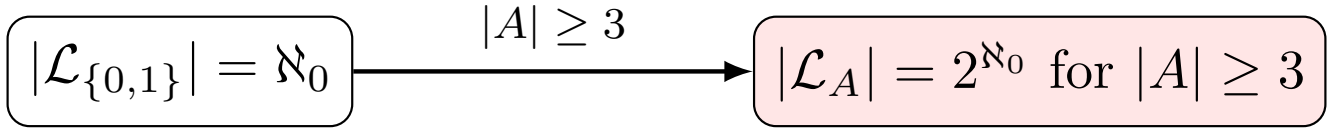


Post's Lattice of Clones on $\{0, 1\}$



The Lattice of Clones on Larger Sets

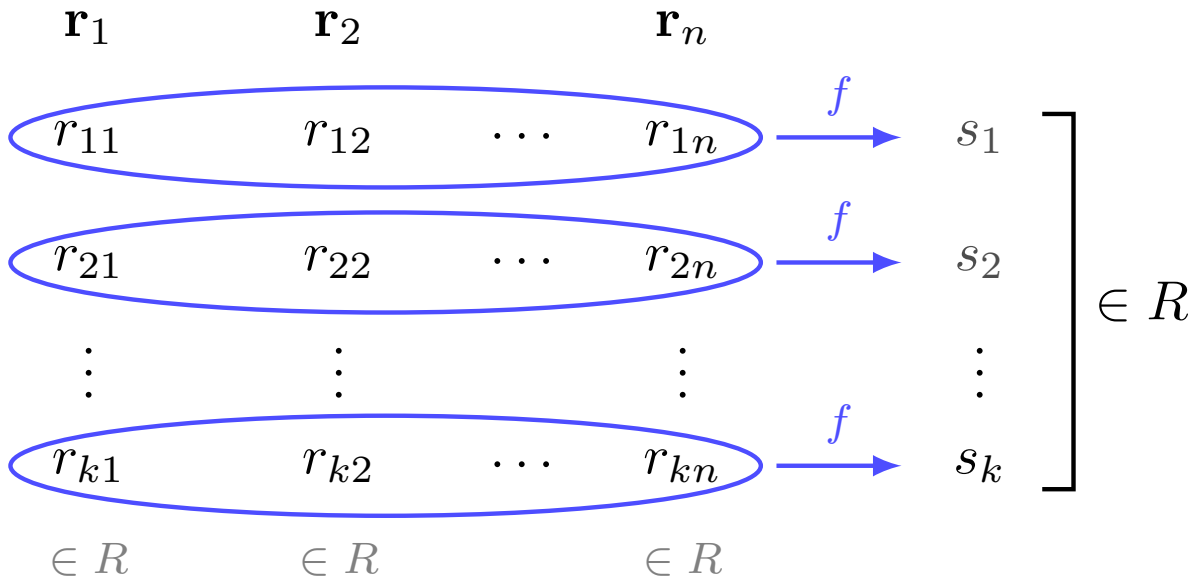
Again for $|A|$ finite, $\mathcal{C} \subseteq \bigcup_{n\in\mathbb{N}} A^{A^n}$ - at most continuum many clones.



- There are **continuum many** clones on any set of size ≥ 3. (Yanov–Muchnik 1959).
- The lattice contains a **rich sublattice structure** – many infinite ascending and descending chains, antichains of size continuum.
- Still just finitely many **maximal clones**
- and finitely many **minimal clones**.

Preserving Relations

A function $f : A^n \to A$ **preserves** a relation $R \subseteq A^k$ if,
$\mathbf{r}_1, \ldots, \mathbf{r}_n \in R \implies$ applying f row by row gives a tuple in R:



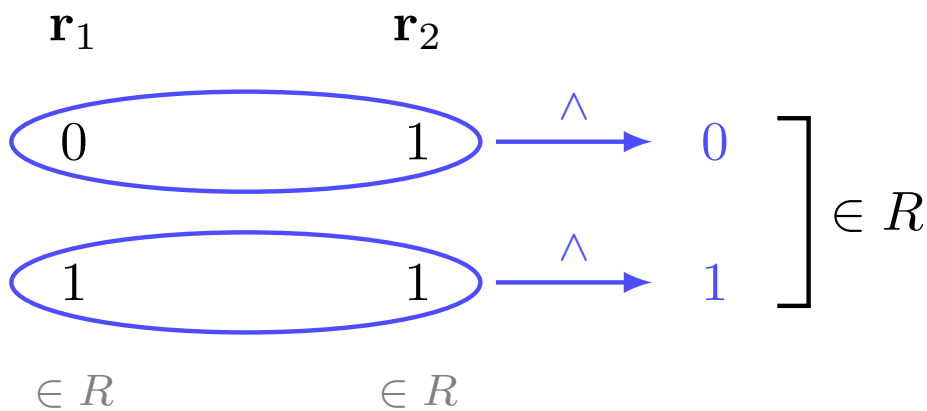
where $s_i = f(r_{i1}, \ldots, r_{in})$ denotes f applied to the i-th row.

Projections preserve every relation.

Preserving Relations: An Example

$A = \{0, 1\}$,
$f(x, y) = x \wedge y$,
$R = \leq = \{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\} \subseteq A^2$.

$\mathbf{r}_1 = \begin{pmatrix}0\\1\end{pmatrix} \in R$ and $\mathbf{r}_2 = \begin{pmatrix}1\\1\end{pmatrix} \in R$. Apply $\wedge$ row by row:



The set of all functions preserving R forms a clone,
the clone of monotone functions on $\{0, 1\}$.

The Pol-Inv Galois Connection

Given a set of relations Q on A, define:

$$\mathrm{Pol}(Q) = \{f : A^n \to A \mid f \text{ preserves every } R \in Q\}$$

$\mathrm{Pol}(Q)$ is a **clone** — closed under composition and contains projections.

In the other direction, given a clone C on A, define:

$$\mathrm{Inv}(C) = \{R \subseteq A^k \mid \text{every } f \in C \text{ preserves } R\}$$

$\mathrm{Inv}(C)$ is a **relational clone**: closed under primitive positive definitions.

These two maps form a **Galois connection**:

$$C \subseteq \mathrm{Pol}(Q) \iff Q \subseteq \mathrm{Inv}(C)$$

The induced closure operators satisfy:

$$\mathrm{Pol}(\mathrm{Inv}(X)) \supseteq X \qquad \mathrm{Inv}(\mathrm{Pol}(Y)) \supseteq Y.$$

Fact: On a finite set, C is a clone $\iff$ $C = \mathrm{Pol}(\mathrm{Inv}(C))$.

Maximal Clones

Theorem (Kuznetsov, 1959)

The maximal clones on A are of the form $\mathrm{Pol}(R)$,
where R is a relation of arity $\leq |A|$.
In particular,

- *there are* ***finitely many*** *maximal clones on a finite set A,*
- *every clone other than $\mathcal{O}_A$ is contained in a maximal clone.*

Rosenberg, 1970: Refined this to 6 families of relations.

Corollary

A set F of operations on A is ***complete*** *— that is,*
F generates $\mathcal{O}_A$ — if and only if

$$F \not\subseteq \mathrm{Pol}(R)$$

for every relation R in these six families.

Minimal Clones

Theorem (Rosenberg, 1986)

Every clone on finite A other than the clone $\mathcal{J}_A$ of projections contains an operation of one of the following five types:

(I) *f is a **unary** operation, $f \notin \mathcal{J}_A$;*

(II) *f is a **binary idempotent** operation, $f \notin \mathcal{J}_A$;*

(III) *f is a **semiprojection** onto its first variable;*

(IV) *f is a **majority** operation;*

(V) *f is a **minority** operation.*

Consequently, a clone $C \supsetneq \mathcal{J}_A$ is minimal if and only if it is generated by a single operation of one of these five types.

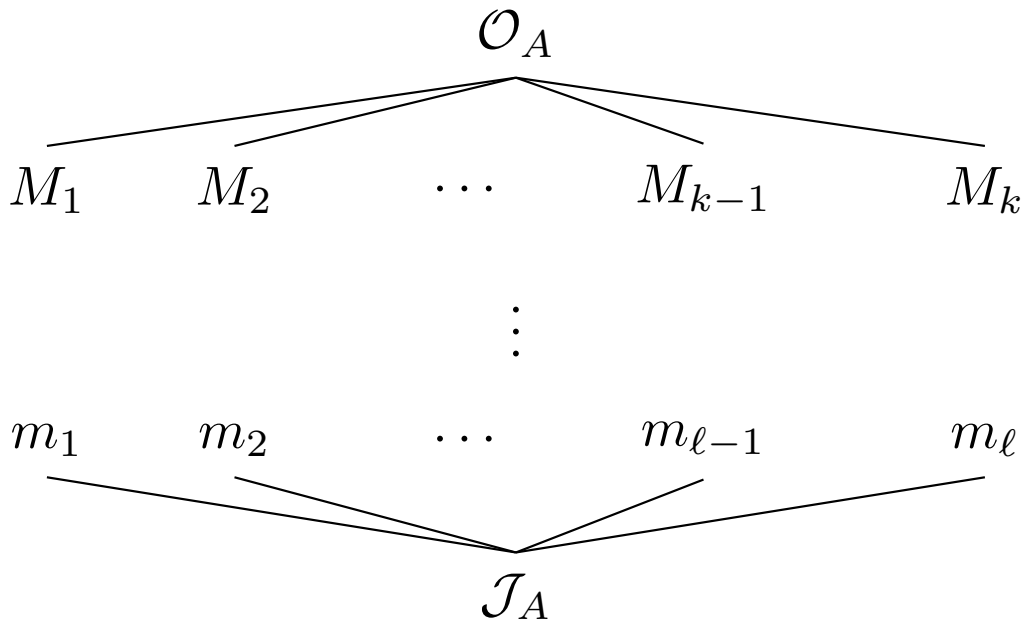
Since there are only finitely many operations of each type...

Theorem

Every clone on finite A (except $\mathcal{J}_A$) has a minimal subclone, and the number of minimal clones on A is finite.

Maximal and Minimal Clones

- There are continuum many clones on finite A with $|A| \geq 3$.
- No hope for a complete classification, but ...
- Top and bottom of the lattice of clones is well-understood.



Part 2: Clonoids

Clonoids

Definition

Let C_1 and C_2 be clones on the sets A and B, respectively.
A set $K \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$ is a (C_1, C_2)**-clonoid** if it is

- closed under precomposition with C_1 (i.e. $K \circ C_1 \subseteq K$), and
- closed under postcomposition with C_2 (i.e. $C_2 \circ K \subseteq K$).

Equivalently, for algebras $\mathbf{A}$ and $\mathbf{B}$ on the sets A and B, resp. an $(\mathbf{A}, \mathbf{B})$**-clonoid** K is a set of functions from A into B with

- $K \circ \mathsf{Clo}(\mathbf{A}) \subseteq K$, and
- $\mathsf{Clo}(\mathbf{B}) \circ K \subseteq K$.

Examples:

- $\mathcal{O}_{A,B}$, the set of all functions from A to B.
- If C is a clone on A, then C is a (C, C)-clonoid.
  - Equivalently, for $\mathbf{A}$ an algebra, $\mathsf{Clo}(\mathbf{A})$ is an $(\mathbf{A}, \mathbf{A})$-clonoid.

More Examples

Example: Let $f\colon A^n \to B$ and $\sigma\colon \{1,\dots,n\} \to \{1,\dots,k\}$.

$$g\colon A^k \to B,\ g(x_1,\dots,x_k) = f(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

is a **minor** of f.
A minion is a set of functions from A to B closed w.r.t minors.
This is a $(\mathcal{J}_A, \mathcal{J}_B)$-clonoid.

Minions play an important role in Constraint Satisfaction Problems.

Example: Let $\mathbf{A}$, $\mathbf{B}$ be abelian groups. The set of homomorphisms

$$K = \bigcup_{n\in\mathbb{N}} \mathrm{Hom}(\mathbf{A}^n, \mathbf{B})$$

forms an $(\mathbf{A}, \mathbf{B})$-clonoid.

Question

Given clones C_1 and C_2 on sets A and B, respectively,
describe the lattice $\mathcal{L}_{(C_1,C_2)}$ of (C_1, C_2)-clonoids.

What is the number of (C_1, C_2)-clonoids?

If A and B are finite then the number of (C_1, C_2)-clonoids
is at most continuum.

When are there continuum many?
countably many?
finitely many?

A Partial Answer

Theorem (Pippenger, 2002)

The number of $(\mathcal{I}_{\{0,1\}}, \mathcal{I}_{\{0,1\}})$*-clonoids is continuum.*

Theorem (Sparks, 2019)

Let A *be finite,* $|A| > 1$*, and let* C *be a clone on* $\{0,1\}$*.*

- $\mathcal{L}_{(\mathcal{I}_A, C)}$ *is finite iff* C *contains a near-unanimity operation.*
- $\mathcal{L}_{(\mathcal{I}_A, C)}$ *is countably infinite iff* C *contains a Mal'cev operation but no majority operation.*
- $\mathcal{L}_{(\mathcal{I}_A, C)}$ *is size continuum iff* C *contains neither a near-unanimity operation nor a Mal'cev operation.*

What if the source clone is not just the projection clone?
What if the target set is of size ≥ 3

Lehtonen (2023-2026) Classified all (C_1, C_2)-clonoids for Boolean clones C_1 and C_2

Clonoids via Relational Pairs

Clones are determined by relations.
Clonoids are determined by **pairs of relations.**

$R \subseteq A^k$, $S \subseteq B^k$. A function $f : A^n \to B$ preserves (R, S) if

$$f(R, R, \ldots, R) \subseteq S.$$

Theorem (Couceiro, Foldes, 2007)

$\mathbf{A}, \mathbf{B}$ *finite algebras.* C *is an* $(\mathbf{A}, \mathbf{B})$*-clonoid iff*

$$C = \bigcap_{i \in I} \mathrm{Pol}(R_i, S_i) \text{ for subalgebras } R_i \leq \mathbf{A}^{n_i}, S_i \leq \mathbf{B}^{n_i}.$$

Theorem (Aichinger, Mayr, 2018)

If $\mathbf{B}$ *is a Mal'cev algebra (group, ring, module, Lie algebra, . . .) and* C *is an* $(\mathbf{A}, \mathbf{B})$*-clonoid then* C *is finitely related.*
Hence $\mathcal{L}_{(\mathbf{A},\mathbf{B})}$ *is finite or countably infinite.*

Clonoids from $\mathbb{Z}_2$ to $\mathbb{Z}_2$

$\mathbf{A} = (\mathbb{Z}_2, +) = \mathbf{B}$, $\quad C$ an $(\mathbf{A}, \mathbf{B})$-clonoid.
Represent $f\colon A^k \to B$ as reduced polynomial, ex:

$$f(x,y,z) = 1 + xz + xyz$$
$$f \in C \implies f(0,0,0) = 1 \in C,$$
$$\& \quad f(x,0,z) = xz + 1 \in C,$$

Hence $\quad xz = (xz+1) - 1 \in C,$
and $\quad xyz = f(x,y,z) - xz - 1 \in C.$

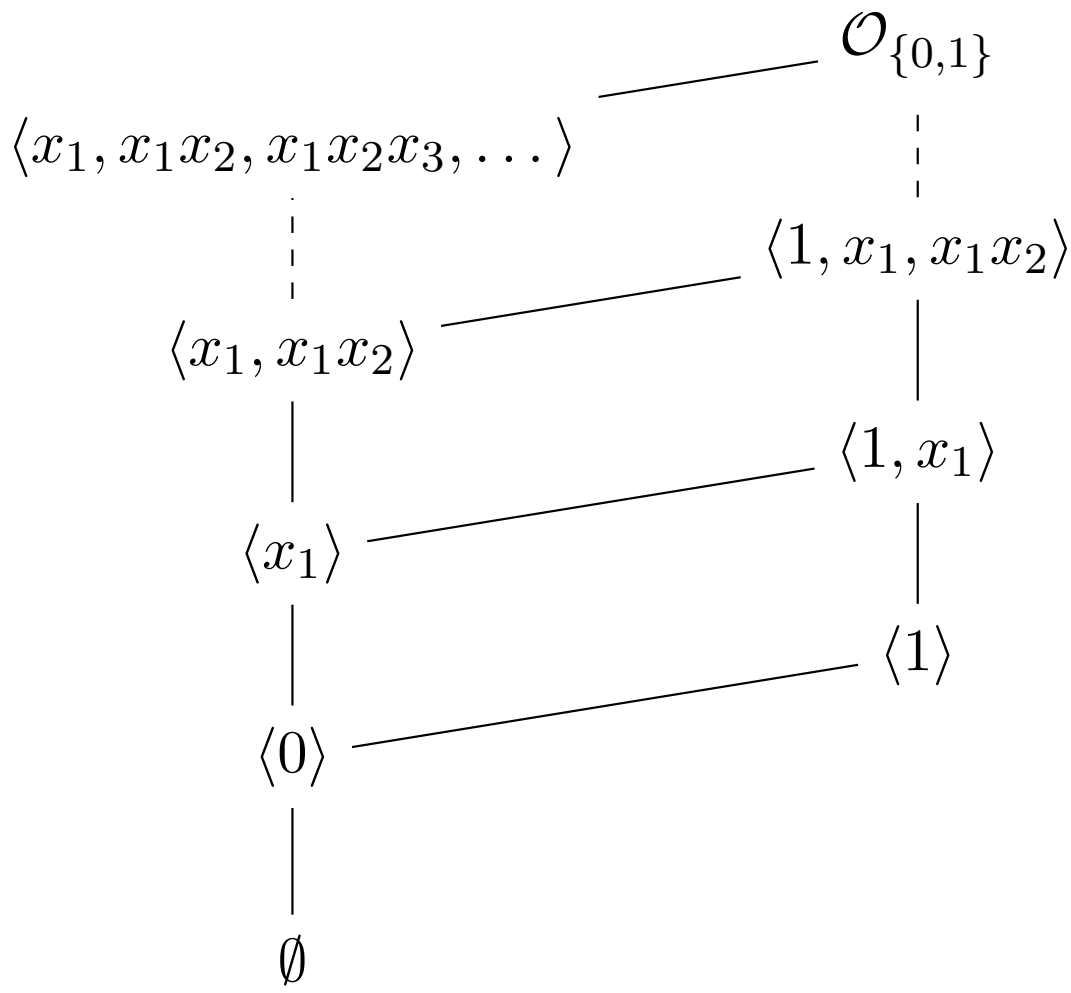
Conversely, if $1 \in C$ and $xyz \in C$ then

$$f(x,y,z) = 1 + xxz + xyz \in C.$$

In general:

$$f \in C \iff f(0,\ldots,0) \in C \ \& \ x_1 \cdot x_2 \cdot \ldots \cdot x_{\deg(f)} \in C.$$

Clonoids from $\mathbb{Z}_2$ to $\mathbb{Z}_2$



Clonoids Between Non-coprime Modules

Theorem (Kreinecker, 2019)

Let $\mathbf{A} = (\mathbb{Z}_p, +) = \mathbf{B}$ *for* p *prime.* $\mathcal{L}(\mathbf{A}, \mathbf{B})$ *is countably infinite. For* $p > 2$*, infinitely many non-finitely generated clonoids.*

Theorem (Mayr, W. 2024)

For $\mathbf{A}$ *and* $\mathbf{B}$ *finite modules with a common prime divisor,* $\mathcal{L}_(\mathbf{A}, \mathbf{B})$ *is countably infinite.*

What if $\mathbf{A}$ and $\mathbf{B}$ are of coprime order?

Clonoids Between Modules of Coprime Order

$$\mathbf{A} = (\mathbb{Z}_2, +), \quad \mathbf{B} = (\mathbb{Z}_3, +), \qquad f : \mathbf{A}^2 \to \mathbf{B}$$

- f is generated by its unary $(\mathbf{A}, \mathbf{B})$-minors.
- Unary minors of f include $f(x, x)$, $f(0, x)$, $f(x, 0)$, $f(0, 0)$.

$$\begin{aligned} f(x, y) = & f(0, 0) \\ & + 2^{-1}[f(x, 0) + f(x + y, 0) - f(0, 0) - f(y, 0) \\ & + f(0, y) + f(0, x + y) - f(0, 0) - f(0, x) \\ & + f(x, x) + f(y, y) - f(0, 0) - f(x + y, x + y)]. \end{aligned}$$

Note that this formula holds independently of the choice of f.

We say $\{f : A^2 \to B\}$ is **uniformly generated** by unary minors.

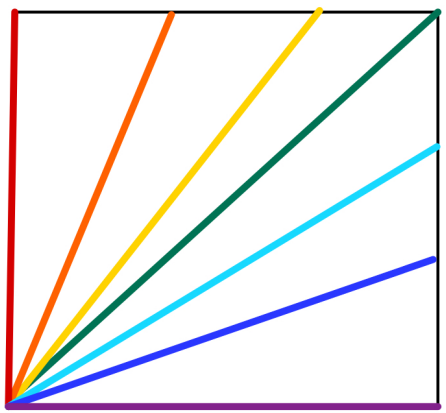
Theorem (Fioravanti, 2020)

Let $\mathbf{A} = (\mathbb{Z}_p, +)$ *and* $\mathbf{B} = (\mathbb{Z}_q, +)$ *for primes* $p \neq q$.
Each $(\mathbf{A}, \mathbf{B})$*-clonoid is uniformly generated by its unary functions.*
Hence, $\mathcal{L}_{(\mathbf{A},\mathbf{B})}$ *is finite.*

$f \colon A^2 \to B, \quad f(0,0) = 0.$

$$f_{L_0}(x,y) := p^{-1}\Big(\sum_{a \in \mathbb{Z}_p} f(x+ay, y) - \sum_{a \in \mathbb{Z}_p} f(ay, y)\Big).$$

$$f_{L_0}(x,y) = \begin{cases} f(x,y) & \text{if } y = 0, \\ 0 & \text{else.} \end{cases}$$



Interpolate on each line through $(0,0)$.

$$f = \textstyle\sum_L f_L.$$

Conjecture: For finite modules $\mathbf{A}, \mathbf{B}$, the lattice of clonoids $\mathcal{L}_{(\mathbf{A},\mathbf{B})}$ is finite if and only if $\mathbf{A}$ and $\mathbf{B}$ are of coprime order.

Some more evidence ...

Theorem (Mayr, W. 2024)

Let $m \in \mathbb{N}$, $\mathbf{A} = (\mathbb{Z}_m; +)$, & $\mathbf{B}$ a module of order coprime to m. Every $(\mathbf{A}, \mathbf{B})$-clonoid is uniformly generated by its n-ary functions, where n is the exponent of $\mathbf{A}$. Hence $\mathcal{L}_{(\mathbf{A},\mathbf{B})}$ is finite.

Holds more generally for $\mathbf{A}$ a distributive module, and n the nilpotence degree of the Jacobson Radical of the ring.

Theorem (Fioravanti, Kompatscher, Rossi, 2026)

Let $\mathbf{A}$ be a k-dimensional vector space over the field $\mathbb{F}_p$, and $\mathbf{B}$ module coprime to p. Every $(\mathbf{A}, \mathbf{B})$-clonoid is generated by its k-ary functions. Hence, $\mathcal{L}_{(\mathbf{A},\mathbf{B})}$ is finite.

Smallest Open Case: $\mathbf{A} = (\mathbb{Z}_4 \times \mathbb{Z}_4; +)$ and $\mathbf{B} = (\mathbb{Z}_3; +)$.

Thank you.